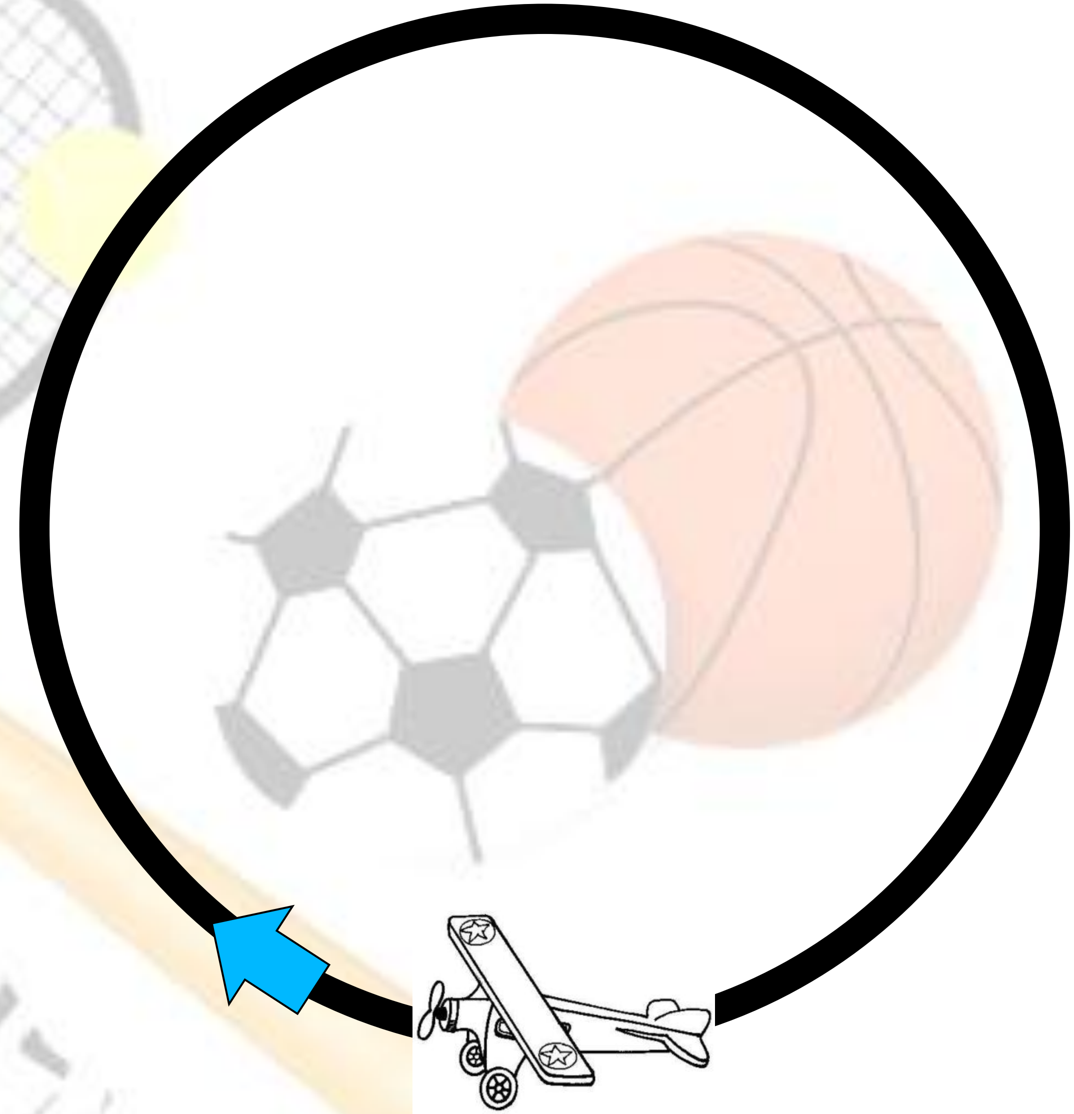
Circular Motion

- Rotational kinetics
- Angular Momentum
- Circular Forces - Torques
- Work energy power in Circular Motion

Rotational Mechanics

- So far we've pretty much confined our study to objects moving in straight lines
- These are 1-D problems
- Important set of problems where objects move in circle
- Called rotational mechanics
- We'll see analogies for speed, acceleration, momentum, force, energy, mass

- Same speed, but in different directions
- Acceleration
- Need to define some quantities before we go any further
- These will be:
  - Period
  - Frequency
  - Angular velocity



Rotational Mechanics – rate of motion

- Periodic time (T)
  - This is the time for one rotation, one lap
- Frequency (f)
  - This is the rate of rotation
  - How many laps per second
  - Units are s^{-1} or Hertz (Hz)
- Angular Velocity (ω, pronounced omega)
  - Angle that the radius sweeps out per unit time
  - Units are radians per second (rads s^{-1})
- Equations: $f=1/T$, $\omega = 2\pi f$

Angular Kinematics

- In the same way that things moving in a straight line can be made to move faster or slower, stuff moving in a circle can rotate faster or slower
- This means a change in angular velocity
- Call this an angular acceleration
- Different from the centripetal acceleration we'll see later
- Use symbol α
  - Units for α are radians per second squared
    - Rad s^{-2}
  - Direction of α are tangent to the circle
    - Parallel or anti-parallel to velocity

Angular Kinematics

- Now have:
  - Angular acceleration, α (note, this is not the same acceleration we met on slide 5)
  - Angular velocity, ω
  - Angular position, θ
  - Put these together to get a set of equations for angular motion:

$$\boldsymbol{\omega = \omega_0 + \alpha t}$$

$$\boldsymbol{\theta = \omega_0 t + \tfrac{1}{2} \alpha t^2}$$

$$\boldsymbol{\omega^2 = \omega_0^2 + 2\alpha\theta}$$

$$\boldsymbol{\theta = (\omega_0 + \omega)t / 2}$$

  - These should look familiar

Angular Kinematics - example

A disc starts from rest and rotates with constant angular acceleration of 2rad s^{-2}. How many revolutions does it make in 10s?

[100rads = 15.92 revolutions]

During a jump, an ice-skater increases their angular velocity from 16 rads s^{-1} to 34.265 rads s^{-1} in 0.5s by tucking in their arms. How many revolutions do they do in this time?

[12.566 rads = 2 revolutions]

After leaving the clubface, a golf ball spins at 346rads s^{-1}. When it lands 7.5s later, its rate of rotation has slowed to 210rads s^{-1}. Calculate the angular deceleration.

[18.13rads s^{-2}]

Angular Kinematics - example

- A discus thrower with a throwing arm length of 1.3m begins his throwing motion with an angular velocity of 5 rads s^{-1}. During the throw he spins through 15 rads. The entire movement takes 0.8s. Calculate the angular acceleration, the final angular velocity, the speed of the discus as it is released, and the tangential acceleration of the discus as it released.

[34.38 rad s^{-2}, 32.5 rads s^{-1}, 42.25 ms^{-1}, 44.69ms^{-2}]

Speed and Acceleration on a Circle

Speed (v)

The speed is distance divided by the time

$$v = \frac{2\pi R}{T} = 2\pi f\ R = \omega R$$

Acceleration (a_r)

Acceleration has units of m/s^2 and the magnitude of the acceleration can be shown to be equal to

$$a_r = v\omega$$

$$a_r = \frac{v^2}{R}$$

$$a_r = \omega^2 R$$

Speed and Acceleration on a Circle Tangential and Radial Acceleration

- There are two types of acceleration in circular motion
  - Radial
  - Tangential
- The radial acceleration is the $a_r = \omega^2 r$ feature we just described
  - Changes direction of object as it moves on the circle
- The tangential acceleration, a_t, is connected to angular acceleration
  - Changes speed of object, faster or slower
- Just like $v = \omega r$ for angular velocity, we get $a_t = \alpha r$

Centrifuge

- The centripetal acceleration of the chamber points radially inwards
- However a particle inside the chamber is not directly subjected to the centripetal force
- It has an acceleration relative to the chamber of $v\omega$ directed outwards
- This is called centrifugal acceleration

- The types of materials placed in a centrifuge are those that do not sediment quickly under the action of gravity
- Centrifugal acceleration plays the role of an artificial gravity acting on the particles suspended in the liquid in the chamber
- Associated with the centrifugal acceleration is the centrifugal force which is IMAGINARY
- It appears as if the particle is being pushed outwards

Moments of Inertia

- In linear momentum we multiplied mass by velocity to get the momentum
- There is an analogous quantity in circular motion
- We'll replace the velocity by the angular velocity
- Need some rotational equivalent for the mass
- Need to factor in that some of the mass in a spinning system is pretty much stationary, stuff near the axis, while some is moving very fast, stuff far from the axis.
- Get *Moment of Inertia* which depends of size of each piece of mass and how far it is from the axis

$$I = \Sigma m r^2$$

Moments of Inertia

- Let's first examine how I depends of the shape of the object
  - For a shell about its axis: $I = MR^2$
  - For a solid cylinder about its axis: $I = \frac{1}{2}MR^2$
  - For a spherical shell about its diameter: $I = \frac{2}{3} MR^2$
  - For a solid ball: $I = \frac{2}{5}MR^2$
  - For a club about one end: $I = \frac{1}{3} MR^2$

Moments of Inertia - examples

- Calculate the moment of inertia of a golf ball, mass = 0.046kg, radius = 2.134cm

[8.38×10^{-6} kgm^2]

- Calculate the moment of inertia of a tennis ball, mass = 0.0567kg, radius = 3.35cm

[4.24×10^{-5} kgm^2]

- Estimate the moment of inertia of a baseball bat, mass = 0.9kg, length = 0.81m

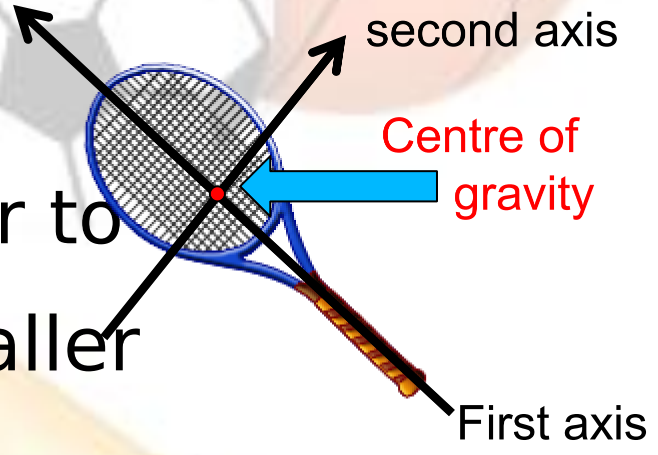
[0.197 kgm^2]

Moments of Inertia - Axes

- With mass, the same object has the same mass every time
- Not so with moment of inertia
- Depends on the axis about which it rotates
- Take human body in standing position:
    - Small I around axis along spine (sagittal)
    - Medium I around axis through midriff from left to right (transverse)
    - Big I around axis coming out through your navel (frontal)

Moments of Inertia - axes

- Same object has different moments of inertia along different axes
- Will depend on average distance of the mass of the object from the axis.
- Take a tennis racket:
  - First axis is longer, it’s closer to most of the mass, has a smaller moment of inertia



- For three dimensional object, cannot spin stably about intermediate axis – Tennis Racket Theorem

Angular Momentum

- When we discussed linear motion we had momentum which was conserved as long as no outside force is acting.
- Of course, get an analogous feature in rotational motion
- Linear momentum was = mass x velocity
- Angular momentum is = (moment of inertia) x (angular velocity)

$$\mathbf{L = I\omega}$$

- And angular momentum is conserved as long as no outside torque is acting

Conservation of Angular Momentum - examples

- High board diver spins at 5 rads s^{-1} about transverse axis in a layout position when her moment of inertia is $15kgm^2$. When she tucks, her moment of inertia decreases to 4 kgm^2. Calculate the new rate of spin.

[18.75 rads s^{-1}]

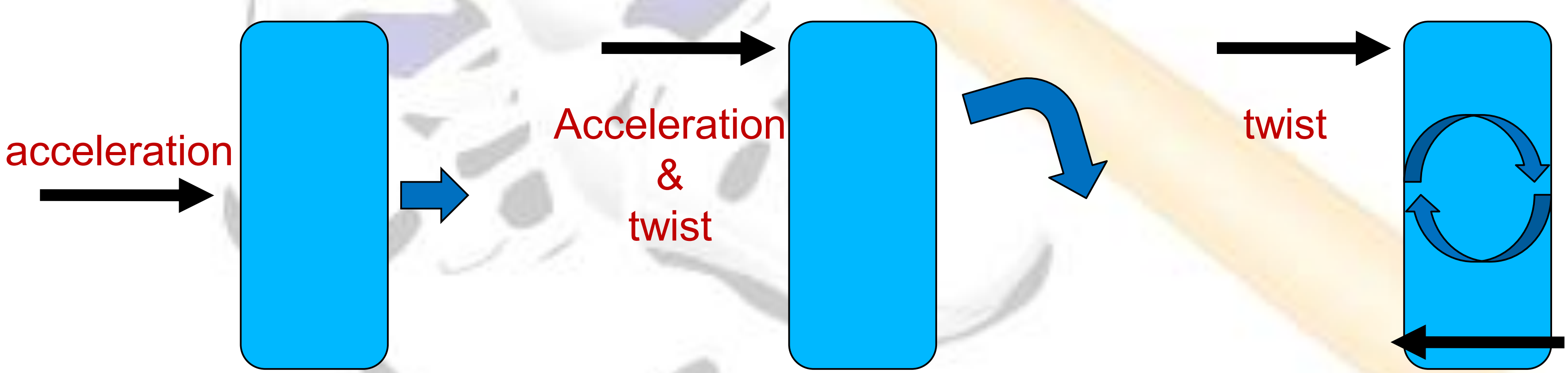
- During a bicycle kick a player swings their leg with an angular momentum of $22kgm^2s^{-1}$ about their hip joint. If the rest of their body has a moment of inertia of $18kgm^2$ about this joint calculate the rate of rotation of their body during the kick. Relative to their kicking leg, in what direction does their body rotate?

Torques

- What causes angular accelerations?
- Need an analogy for the force we used in linear motion
- This change in angular velocity produced by a ***Torque***

Torques - definition

- Push middle of an object – it accelerates
- Push edge of an object – it accelerates and turns
- Push in opposite directions at top and bottom – it turns



Torques - definition

- Making something rotate depends on:
  - How hard you push
  - How far from centre you push
- Concept of *lever arm*
- Magnitude of torque is defined to be:

Torque = Force x Distance

- The distance is measured from the line of action of the force to the hinge, the centre of rotation, of the object

Torques - definition

- Torques can be positive or negative
  - Torque is a vector
- By convention, torques causing counter-clockwise rotations are positive
- Torques can add, subtract, cancel each other out
- Units for torque are Newtons x metres = Nm

(bizarrely this is the same for work and energy, but they are totally different things)

Torques - example

- When jumping a person exerts a force through their Achilles tendon of 850N. The tendon is attached 3.5cm from the ankle joint. Calculate the torque about the joint.

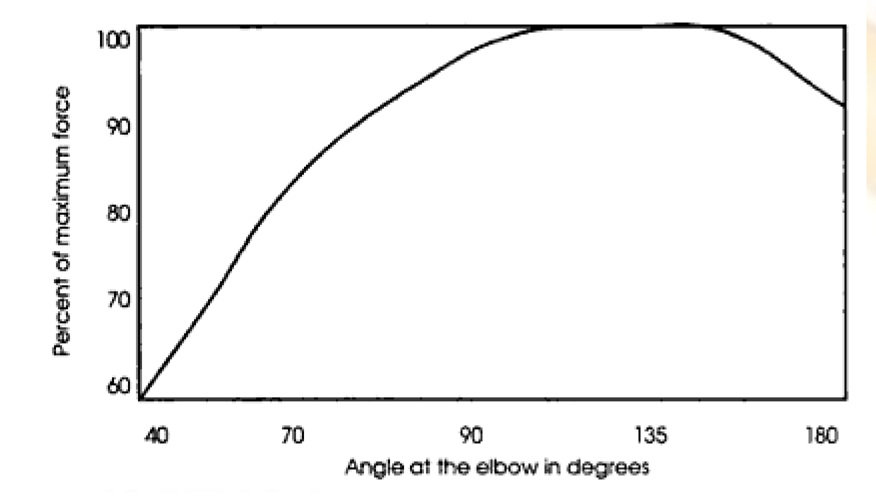
[29.75Nm]

Torques in Sport

- Any sporting action that involves a turn, spin, or swing
  - e.g rowing, golf, tennis, cricket
- Feature of tackling and wrestling
- Principle behind how most weights machines work

Torques in Sport

- Also important to understand muscle action
  - See previous example
  - Different torques as a joint is flexed
  - Maximum torque for elbow at a right angle
  - Weight machines designed to match this torque curve (dynamic variable resistance)



Torques in Muscles

action	Peak muscle torque (Nm)
Trunk extension	258
Trunk flexion	177
Knee extension	204
Knee flexion	109
Hip extension	150
Ankle plantar flexion	74
Elbow flexion	20
Wrist flexion	8
Wrist extension	4

(from Knudson, Biomechanics, p.171)

Torque and Moments of Inertia - examples

- A string is wound around a disk of mass 3kg and radius 0.25m. The string is pulled with a force of 10N. The disk is intially at rest
  - What is the moment of inertia of the disk?
  - What is the torque on the disk?
  - What is the angular acceleration of the disk?
  - what is the angular velocity after 7s?

[0.0938kgm^2, 2.5Nm, 26.7rads s^{-2}, 186.7rads s^{-1}]

Torque and Moments of Inertia - examples

- A 2.9kg mass is attached to a string wound around a wheel of mass 6.1kg and radius 0.75m.
  - Calculate the moment of inertia of the wheel
  - Calculate the weight of the attached mass
  - Calculate the acceleration of the attached mass as it falls

[1.716kgm^2, 28.45N, 4.78ms^{-2}]

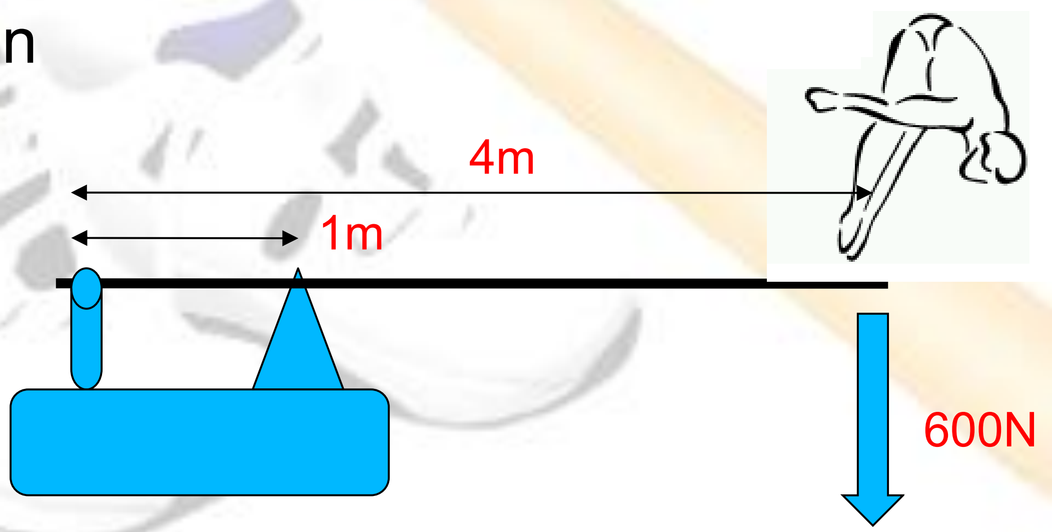
Moments of Inertia - planes

- To identify the axis of rotation it can help to think about the plane of rotation
- If you can picture a two dimensional sheet in which a part of the object is moving, then the axis of rotation is perpendicular to this sheet.
- **Sagittal Plane** – divides body left and right
- **Coronal Plane** – divides body back and front
- **Transverse Plane** – divides body top and bottom
- **Transverse Axis** – perpendicular to sagittal plane

- **Anteroposterior Axis** – perpendicular to coronal

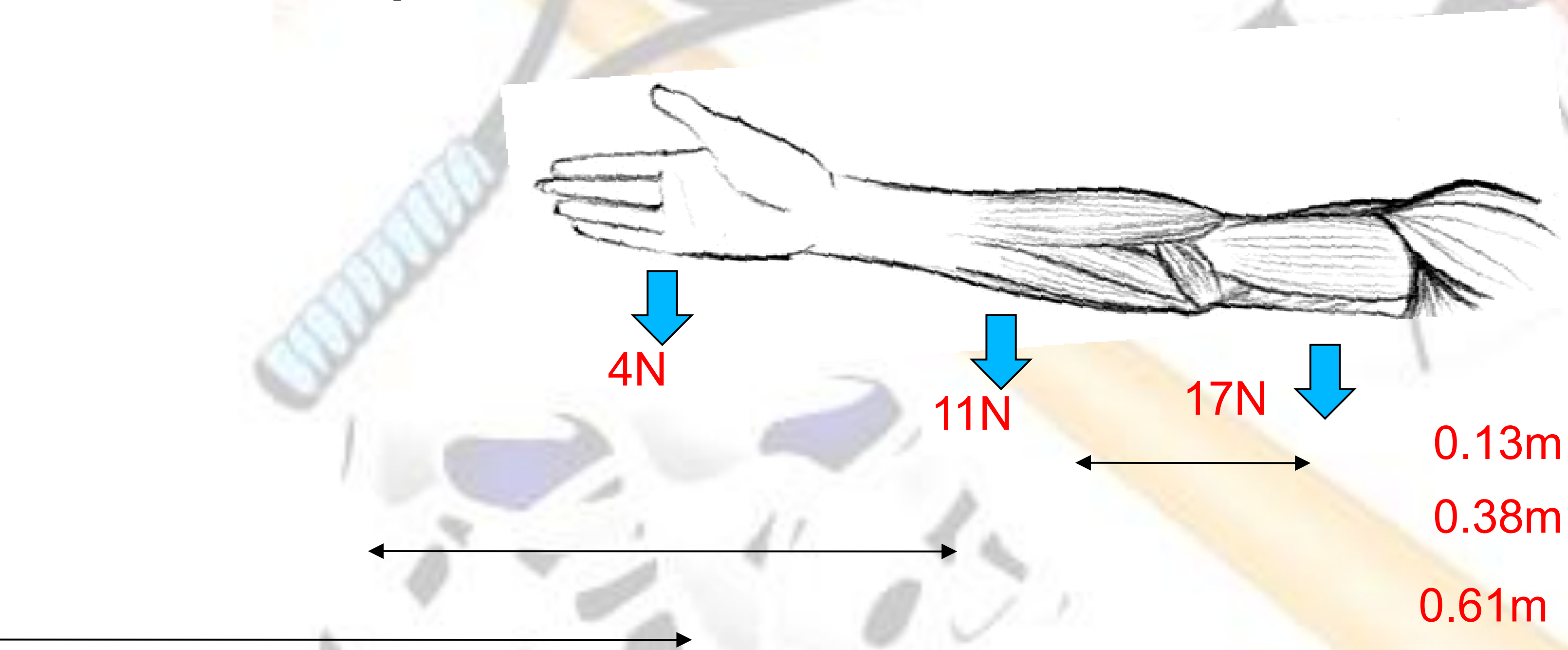
Torques at Equilibrium

- As torques are vectors they can add and subtract
- Can get a force couple that gives us no net torque
- This is the law of the lever
- Consider the diver on a spring board below:
  - Can figure out that the force on the bolt is 1800N and on the fulcrum is 2400N
  - Note that in equilibrium, it doesn't matter where we choose the axis of rotation



Centre of Gravity

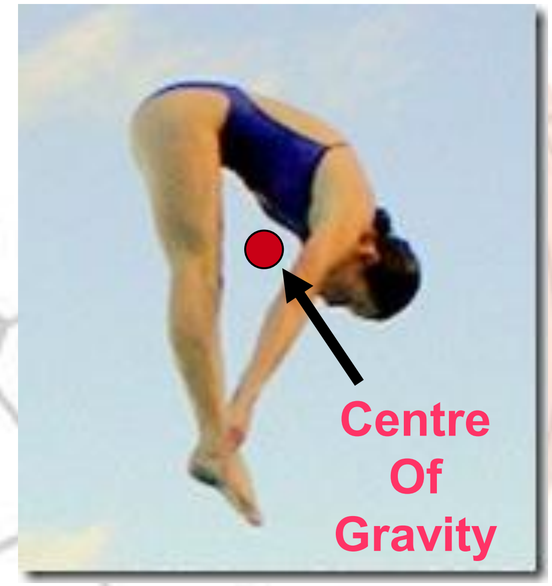
- Point at which entire mass of a body seems to be concentrated
- Example: consider a human arm

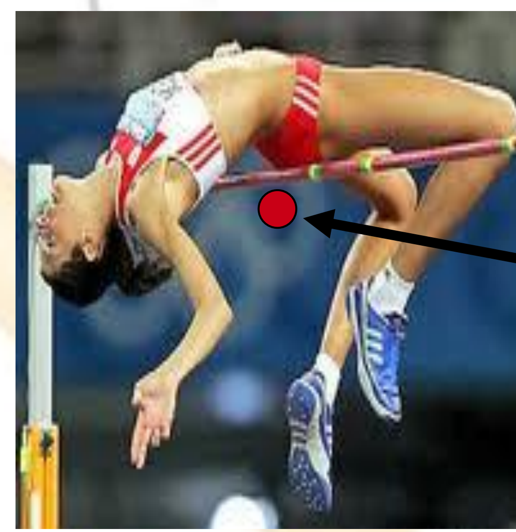


$$x_{cog} = \frac{17 \times 0.13 + 11 \times 0.38 + 4 \times 0.61}{17 + 11 + 4} = 0.276m$$

Centre of Gravity

- In humans, COG depends on position of limbs
- In normal resting position (arms by your side), your COG is slightly below your navel
  - About 55% of body height for women
  - About 57% of body height for men
- If limbs move COG changes
- Can be outside body
  - Highjumper clearing bar, COG below backbone
  - Springboard diver in piked position, COG between limbs





Centre of Gravity

- Lots of sporting techniques involve adjustments of the COG
- Example is high jump from previous slide
- In other sports players can seem to hang in the air
  - COG follows parabola, as it must
  - Movement of limbs means other parts of body can be stable in the air
  - Example; dropping your non-striking arm in volleyball gives steadier platform from which to spike
- Easier to reach higher with one arm raised rather than two

Measuring Centre of Gravity

- Two methods
  - Suspend object twice from different points of support
    - Intersection of vertical lines from points of support is COG
  - Measure total mass of object then measure mass with one end supported
    - Fractional decrease in measured mass tells you how far up object COG is

Stability

- Ability to return to balance, to equilibrium, after having been displaced
- Sometimes stability desirable
  - Golfers
  - Wrestlers
- Sometimes not
  - Goalie
  - Serve receive

Stability

- Factors affecting stability
  - Size of base of support
    - The bigger the more stable
  - Height of centre of gravity
    - Closer to support the more stable
  - Weight of object
    - Greater weight means more stable
  - Torque of force trying to disturb equilibrium
    - Greater the torque, the further from COG, the greater the chance of upsetting stability
- Most stable stance is the one which minimises your potential energy

Stability - Example

- Car negotiating a turn. Given the wheelbase of the car, the height of the COG, and the radius of curvature of the corner, calculate the maximum speed the car can travel without overturning.
  - Torque produced by centrapetal force of circular motion ($a=v^2/r$)
  - This is counteracted by weight of car
  - End up with

$$v_{max} = \sqrt{\frac{rgd}{2h}}$$

Moments of Inertia - planes

- Human body’s moment of inertia is largest around anteroposterior axis, smallest around longitudinal axis
- By tennis racket theorem, rotation around transverse axis unstable
- Typical moments of inertia are 10’s of kgm^2
- Depends on axis and depends on body position

Moments of Inertia – human movement

- Terms describing movement of:
  - Elbow
  - Knee
  - Shoulder
  - Hip
  - Wrist
  - Ankle

Moments of Inertia – human movement

- In sagittal plane
  - **Flexion** –around transverse axis, away from rest position
  - **Extension** – opposite of flexion, returns to rest position
  - **Dorsiflexion** – flexion of the ankle joint
  - **Plantar Flexion** – extension for the ankle (standing on your toes is plantar flexion)
- In frontal plane
  - **Abduction** – around anteroposterior axis, away from rest position
  - **Adduction** – opposite of abduction, back to rest position

Rotational Kinetic Energy

- Spinning object has energy
- Depends on rate of rotation and of mass distribution
- Get Rotational Kinetic Energy = RKE = $\frac{1}{2} I\omega^2$

Rotational Kinetic Energy - examples

- Re-examine the problem on slide 29 and solve using energy conservation. Note how much easier it is to do this way.
- A bowling ball has a radius of 0.269m, a mass of 6.5kg, and a moment of inertia of 0.188kgm^2. Its speed is 5ms^{-1}. Calculate its kinetic energy, its angular velocity, and its angular kinetic energy.

[81.25J, 18.6rads s^{-1}, 32.52J]

https://ciechanow.ski/bicycle/

Biomechanics Summary

- Period / frequency
- Radians / angular velocity
- Angular kinetics
- Angular momentum, conservation
- Moment of inertia
- Torques
- Rotational kinetic energy

Biomechanics Summary

- $T = 1/f = 2\pi/\omega$
- $F = 1/T = \omega/2\pi$
- $\omega = 2\pi f = 2\pi/T$
- $v = \omega R$: $a_t = \alpha R$
- $a_r = v\omega = v^2/R = \omega^2 R$
- τ = force x distance

- $\omega = \omega_0 + \alpha t$
- $\theta = \omega_0 t + \frac{1}{2} \alpha t^2$
- $\omega^2 = \omega_0^2 + 2\alpha\theta$
- $\theta = (\omega_0 + \omega)t / 2$
- $I = \Sigma mR^2$
- $\tau = I\alpha$
- $RKE = \frac{1}{2} I\omega^2$
- $L = I\omega$
- $L_{before} = L_{after}$